Optimal Sequential Frame Synchronization

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Abstract

We consider the 'one-shot frame synchronization problem' where a decoder wants to locate a sync pattern at the output of a channel on the basis of sequential observations. We assume that the sync pattern of length N starts being emitted at a random time within some interval of size A, that characterizes the asynchronism level between the transmitter and the receiver. We show that a sequential decoder can optimally locate the sync pattern, i.e., exactly, without delay, and with probability approaching one as $N \to \infty$, if and only if the asynchronism level grows as $O(e^{N\alpha})$, with α below the *synchronization threshold*, a constant that admits a simple expression depending on the channel. This constant is the same as the one that characterizes the limit for reliable asynchronous communication, as was recently reported by the authors. If α exceeds the synchronization threshold, any decoder, sequential or non-sequential, locates the sync pattern with an error that tends to one as $N \to \infty$. Hence, a sequential decoder can locate a sync pattern as well as the (non-sequential) maximum likelihood decoder that operates on the basis of output sequences of maximum length $A + N - 1$, but with much fewer observations.

Index Terms

Quickest detection, frame synchronization, sequential analysis

I. Introduction

Frame synchronization refers to the problem of locating a sync pattern periodically embedded into data and received over a channel (see, e.g., [4], [3], [6], [5]). In [4] Massey considered the situation of binary data transmitted across a white Gaussian noise channel. He showed that, given received data of fixed size which the sync pattern is known to belong to, the maximum likelihood rule consists of selecting the location that maximizes the sum of the correlation and a correction term.

We are interested in the situation where the receiver wants to locate the sync pattern on the basis of sequential observations, which Massey refers to as the 'one-shot' frame synchronization problem in [4]. Surprisingly, this setting has received much less attention than the fixed length frame setting. In particular it seems that this problem hasn't been given a precise formulation yet. In this note we propose a formulation where the decoder has to locate the sync pattern exactly and without delay, with the foreknowledge that the pattern is sent within a certain time interval that characterizes the level of asynchronism between the transmitter and the receiver. Our main result is the asymptotic characterization of the largest asynchronism level with respect to the size of the sync pattern for which a decoder can correctly perform with arbitrarily high probability.

II. Problem formulation and result

We consider discrete-time communication over a discrete memoryless channel characterized by its finite input and output alphabets $\mathcal{X}$ and $\mathcal{Y}$, respectively, transition probability matrix $Q(y|x)$, for all $y \in \mathcal{Y}$ and $x \in \mathcal{X}$, and 'noise' symbol $\star \in \mathcal{X}$.[1]

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[1]Throughout this note we always assume that for all $y \in \mathcal{Y}$ there is some $x \in \mathcal{X}$ for which $Q(y|x) > 0$.

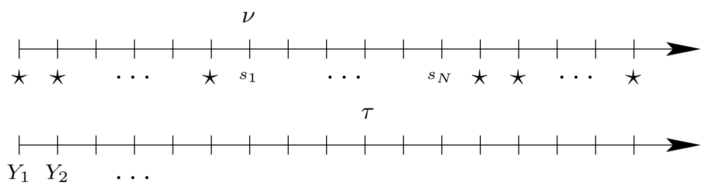

Fig. 1. Time representation of what is sent (upper arrow) and what is received (lower arrow). The '$\star$' represents the 'noise' symbol. At time ν the sync pattern starts being sent and is detected at time τ.

The sync pattern s^N consists of $N \geq 1$ symbols from $\mathcal{X}$ — possibly also the $\star$ symbol. The transmission of the sync pattern starts at a random time ν, uniformly distributed in $[1, 2, \ldots, A]$, where the integer $A \geq 1$ characterizes the asynchronism level between the transmitter and the receiver.

We assume that the receiver knows A but not ν. Before and after the transmission of the sync pattern, i.e., before time ν and after time $\nu + N - 1$, the receiver observes noise. Specifically, conditioned on the value of ν, the receiver observes independent symbols $Y_1, Y_2, \ldots$ distributed as follows. If $i \leq \nu - 1$ or $i \geq \nu + N$, the distribution is $Q(\cdot|\star)$. At any time $i \in [\nu, \nu + 1, \ldots, \nu + N - 1]$ the distribution is $Q(\cdot|s_{i-\nu+1})$, where s_n denotes the nth symbol of s^N.

To identify the instant when the sync pattern starts being emitted, the receiver uses a sequential decoder in the form of a stopping time τ with respect to the output sequence $Y_1, Y_2, \ldots$[2] If $\tau = n$ the receiver declares that the sync pattern started being sent at time $n - N + 1$ (see Fig. 1). The associated error probability is defined as

$$\mathbb{P}(\tau \neq \nu + N - 1) \; .$$

We now define the *synchronization threshold*.

Definition. *An* asynchronism exponent α is achievable *if there exists a sequence of pairs sync pattern/decoder* $\{(s^N, \tau_N)\}_{N \geq 1}$ *such that* s^N *and* τ_N *operate under asynchronism level* $A = e^{\alpha N}$*, and so that*

$$\mathbb{P}(\tau_N \neq \nu - N + 1) \stackrel{N \to \infty}{\longrightarrow} 0 \; .$$

The synchronization threshold, *denoted* $\alpha(Q)$*, is the supremum of the set of achievable asynchronism exponents.*

Our main result lies in the following theorem.

Theorem. *The synchronization threshold as defined above is given by*

$$\alpha(Q) = \max_x D(Q(\cdot|x)||Q(\cdot|\star)) \; ,$$

where $D(Q(\cdot|x)||Q(\cdot|\star))$ *is the divergence (Kullback-Leibler distance) between* $Q(\cdot|x)$ *and* $Q(\cdot|\star)$*. Furthermore, if the asynchronism exponent is above the synchronization threshold, a maximum likelihood decoder that is revealed the maximum length sequence of size* $A+N-1$ *makes an error with a probability that tends to one as* $N \to \infty$*.*

A direct consequence of the theorem is that a sequential decoder can (asymptotically) locate the sync pattern as well as the optimal maximum likelihood decoder that operates on a non-sequential basis, having access to sequences of maximum size $A + N - 1$.[3]

Note that the synchronization threshold is the same as the one in [7], which is defined as the largest asynchronism level for which reliable communication can be achieved over point-to-point asynchronous

[2] Recall that a stopping time τ is an integer-valued random variable with respect to a sequence of random variables $\{Y_i\}_{i=1}^{\infty}$ so that the event $\{\tau = n\}$, conditioned on $\{Y_i\}_{i=1}^{n}$, is independent of $\{Y_i\}_{i=n+1}^{\infty}$ for all $n \geq 1$.

[3] Note that if the receiver has no foreknowledge on A, i.e., if A can a priori be arbitrarily large, the problem is ill-posed: for any decoder, the probability of miss-location of the sync pattern can be made arbitrarily large for A large enough.

channels. This should not come as a surprise since the limit of asynchronous communication is obtained in the zero rate regime where decoding errors are mainly due to a miss location of the transmitted message.

We now prove the theorem by first presenting the direct part and then its converse. Recall that a type, or empirical distribution, induced by a sequence $z^N \in \mathcal{Z}^N$ is the probability $\hat{P}$ on $\mathcal{Z}$ where $\hat{P}(a)$, $a \in \mathcal{Z}$, is equal to the number of occurrences of a in z^N divided by N.

Proof of achievability: We show that a suitable sync pattern together with the sequential typicality decoder[4] achieves an asynchronism exponent arbitrarily close to the synchronization threshold. The intuition is as follows. Let $\bar{x}$ be a 'maximally divergent symbol,' i.e., so that

$$D(Q(\cdot|\bar{x})||Q(\cdot|\star)) = \alpha(Q) \ .$$

Suppose the sync pattern consists of N repetitions of $\bar{x}$. If we use the sequential typicality decoding we already have almost all the properties we need. Indeed, if $\alpha < \alpha(Q)$, with negligible probability the noise generates a block of N output symbols that is jointly typical with the sync pattern. Similarly, the block of output symbols generated by the sync pattern is jointly typical with the sync pattern with high probability. The only problem occurs when a block of N output symbols is generated partly by noise and partly by the sync pattern. Indeed, consider for instance the block of N output symbols from time $\nu - 1$ up to $\nu + N - 2$. These symbols are all generated according to the sync pattern, except for the first. Hence, whenever the decoder observes this portion of symbols, it makes an error with constant probability. The argument extends to any fixed length shift.

The reason that the decoder is unable to locate the sync pattern exactly is that a constant sync pattern has the undesirable property that when it is shifted to the right, it still looks almost the same. Therefore, to prove the direct part of the theorem, we consider a sync pattern mainly composed of $\bar{x}$'s, but with a few $\star$'s mixed in[5] to make sure that shifts of the sync pattern look sufficiently different from the original sync pattern. This allows the decoder to identify the sync pattern exactly, with no delay, and with probability tending to one as N goes to infinity, for any asynchronism exponent less than $\alpha(Q)$. We formalize this below.

Suppose that, for any arbitrarily large K, we can construct a sequence of patterns $\{s^N\}$ of increasing lengths such that each $s^N = s_1, s_2, \ldots, s_N$ satisfies the following two properties:

I. all s_i's are equal to $\bar{x}$, except for a fraction smaller than $1/K$ that are equal to $\star$;

II. the Hamming distance between the pattern and any of its shifts of the form

$$\underbrace{\star, \star, \ldots, \star}_{i \text{ times}}, s_1, s_2, \ldots, s_{N-i} \qquad i \in [1, 2, \ldots, N]$$

is linear in N.

Now let $A = e^{N(\alpha(Q)-\epsilon)}$, for some $\epsilon > 0$, and consider using patterns with the properties I and II in conjunction with the sequential typicality decoder $\tau = \tau_N$.

By [1, Lemma 2.6, p.32] and property I, the probability that N output symbols entirely generated by noise are typical with the sync pattern is upper bounded by $\exp\left(-N(1-1/K)(\alpha(Q)-\delta)\right)$, where $\delta > 0$ goes to zero as the typicality constant μ goes to zero.[6] Hence, by the union bound

$$\mathbb{P}\left(\{\tau < \nu\} \cup \{\tau \geq \nu + 2N - 1\}\right) \leq e^{-N(\epsilon - \delta - (\alpha - \delta)/K)}$$

which tends to zero for μ small enough and K sufficiently large.[7]

[4]The sequential typicality decoder operates as follows. At time n, it computes the empirical distribution $\hat{P}$ induced by the sync pattern and the previous N output symbols $y_{n-N+1}, y_{n-N+2}, \ldots, y_n$. If this distribution is close enough to P, i.e., if $|\hat{P}(x,y) - P(x,y)| \leq \mu$ for all x, y, the decoder stops, and declares $n - N + 1$ as the time the sync pattern started being emitted. Otherwise it moves one step ahead and repeats the procedure. Throughout the argument we assume that μ is a negligible strictly positive quantity.

[5]Indeed, any symbol different than $\bar{x}$ can be used.

[6]See footnote 4.

[7]If $\alpha(Q) = \infty$ the upper bound is zero if μ is small enough.

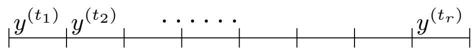

Fig. 2. Parsing of the entire received sequence of size $A+N-1$ into r blocks $y^{(t_1)}, y^{(t_2)}, \ldots, y^{(t_r)}$ of length N, where the ith block starts at time t_i.

If the N observed symbols are partly generated by noise and partly by the sync pattern, by property II, the Chernoff bound, and the union bound we obtain

$$\mathbb{P}\left(\tau \in [\nu, \nu+1, \ldots, \nu+N-2]\right) \leq (N-1)e^{-\Omega(N)}$$

which vanishes as N tends to infinity.

We then deduce that

$$\mathbb{P}(\tau = \nu + N - 1) \to 1$$

as $N \to \infty$.

To conclude we give an explicit construction of a sequence of sync pattern satisfying the properties I and II above. To that aim we use maximal length shift register sequences (see, e.g.,[2]). Actually, for our purpose, the only property we use from such binary sequences of length $l = 2^m - 1$, $m \in [1, 2, \ldots)$, is that they are of Hamming distance $(l+1)/2$ from any of their circular shifts.

To construct the sync pattern we start by setting $s_i = \bar{x}$ for all $i \not\equiv 0 \bmod K$ where, without loss of generality, K is chosen to satisfy $\lfloor \frac{N}{K} \rfloor = 2^m - 1$ for some $m \in [1, 2, \ldots)$.[8] With this choice, property I is already satisfied. To specify the $\lfloor \frac{N}{K} \rfloor$ positions i with $i \equiv 0 \bmod K$, pick a maximal length shift register sequence $m_1, m_2, \ldots, m_{\lfloor \frac{N}{K} \rfloor}$, and set $s_{jK} = \bar{x}$ if $m_j = 0$ and $s_{jK} = \star$ if $m_j = 1$, for any integer $j \leq \lfloor \frac{N}{K} \rfloor$. It can be readily verified, using the circular shift property of maximal length shift register sequences, that this construction yields patterns that satisfy property II.

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Proof of the converse: We assume that $A = e^{N\alpha}$ with

$$\alpha > \max_x D(Q(\cdot|x)||Q(\cdot|\star))$$

and show that the (optimal) maximum likelihood decoder that operates on the basis of sequences of maximum length $A+N-1$ yields a probability of error going to one as N tends to infinity.

We assume that the sync pattern s^N is composed of N identical symbols $s \in \mathcal{X}$. The case with multiple symbols is obtained by a straightforward extension. Suppose the maximum likelihood decoder not only is revealed the complete sequence

$$y_1, y_2, \ldots, y_{A+N-1} \ ,$$

but also knows that the sync pattern was sent in one of the r distinct block periods of duration N, where r denotes the integer part of $(A+N-1)/N$, as shown in Fig. 2.

Assuming $Q(y|\star) > 0$ for all $y \in \mathcal{Y}$,[9] straightforward algebra shows that the decoder outputs the time t_i, $i \in [1, 2, \ldots, r]$, that maximizes

$$f(y^{(t_i)}) = \frac{Q(y^{(t_i)}|s^N)}{Q(y^{(t_i)}|\star)} \ .$$

Note that $f(y^{(t)})$ depends only on the type of the sequence $y^{(t)}$ since s^N is the repetition of a single symbol. For conciseness, from now on we adopt the notation $Q_s(y^{(t)})$ instead of $Q(y^{(t)}|s^N)$ and $Q_\star(y^{(t)})$ instead of $Q(y^{(t)}|\star)$.

[8] We use $\lfloor x \rfloor$ to denote the largest integer smaller than x.

[9] If $Q(y|\star) = 0$ for some $y \in \mathcal{Y}$ we have $\alpha(Q) = \infty$, and there is nothing to prove.

Let $Q_s \pm \varepsilon_0$ denote the set of types (induced by sequences y^N) that are $\varepsilon_0 > 0$ close to Q_s with respect to the L_1 norm, and let E_1 denote the event that the type of the νth block (corresponding to the sync transmission period) is not in $Q_s \pm \varepsilon_0$. It follows that

$$\mathbb{P}(E_1) \leq e^{-N\epsilon} \tag{1}$$

for some $\varepsilon = \varepsilon(\varepsilon_0) > 0$.[10] Let $\bar{Q}_s = \arg\max_{P \in Q_s \pm \epsilon_0} f(P)$,[11] where with a slight abuse of notation $f(P)$ is used to denote $f(y^N)$ for any sequence y^N having type P. Now consider the event E_2 where the number of blocks generated by $Q_\star$ that have type $\bar{Q}_s$ is smaller than

$$\frac{1}{2(N+1)^{|\mathcal{X}|}} e^{-N(D(\bar{Q}_s||Q_\star)-\alpha)} \, .$$

Using [1, Lemma 2.6, p.32], the expected number of blocks generated by $Q_\star$ that have type $\bar{Q}_s$ is lower bounded as

$$\begin{aligned} \mathbb{E}\left(\text{number of type } \bar{Q}_s \text{ blocks generated from } Q_\star\right) &\geq \frac{1}{(N+1)^{|\mathcal{X}|}} e^{-ND(\bar{Q}_s||Q_\star)}(r-1) \\ &\geq \text{poly}(N) e^{-N(D(\bar{Q}_s||Q_\star)-\alpha)} \, , \end{aligned}$$

and using Chebyshev's inequality we get

$$\mathbb{P}\big(E_2\big) \leq \text{poly}(N) e^{-N(\alpha - D(\bar{Q}_s||Q_\star))} \tag{2}$$

where $\text{poly}(N)$ denotes a term that increases or decreases no faster than polynomially in N.

Finally consider the event E_3 defined as the complement of $E_1 \cup E_2$. Given that E_3 happens, the decoder sees at least

$$\frac{1}{2(N+1)^{|\mathcal{X}|}} e^{-N(D(\bar{Q}_s||Q_\star)-\alpha)}$$

time slots that are at least as probable as the correct νth. Hence, the probability of correct detection given that the event E_3 happens is upper bounded as

$$\mathbb{P}(\text{corr.dec}|E_3) \leq \text{poly}(N) e^{-N(\alpha - D(\bar{Q}_s||Q_\star))} \, . \tag{3}$$

We deduce from (1), (2), and (3) that the probability of correct decoding is upper bounded as

$$\begin{aligned} \mathbb{P}\left(\text{corr. dec.}\right) &= \sum_{i=1}^{3} \mathbb{P}(\text{corr.dec}|E_i)\mathbb{P}(E_i) \\ &\leq \mathbb{P}(E_1) + \mathbb{P}(E_2) + \mathbb{P}(\text{corr.dec}|E_3) \\ &\leq (e^{-N\varepsilon} + e^{-N(\alpha - D(\bar{Q}_s||Q_\star))}) \text{poly}(N) \, . \end{aligned}$$

Therefore if

$$\alpha > D(\bar{Q}_s||Q_\star) \, ,$$

the probability of successful detection goes to zero as N tends to infinity. Since $D(\bar{Q}_s||Q_\star)$ tends to $D(Q_s||Q_\star)$ as $\varepsilon_0 \downarrow 0$ by continuity of $D(\cdot||Q_\star)$,[12] the result follows by maximizing $D(Q_s||Q(\cdot|\star))$ over $s \in \mathcal{X}$.

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[10] Here we implicitly assume that N is large enough so that the set of types $Q_s \pm \varepsilon_0$ is nonempty.

[11] Note that $\bar{Q}_s$ may not be equal to Q_s.

[12] We may assume $D(\cdot||Q_\star)$ is continuous because otherwise $\alpha(Q) = \infty$ and there is nothing to prove.

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